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Generalized Supergravity in Two Dimensions

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Abstract

Among the usual constraints of (1,1) supergravity in $d = 2$ the condition of vanishing bosonic torsion is dropped. Using the *inverse* supervierbein and the superconnection considerably simplifies the formidable computational problems. It allows to solve the constraints for those fields *before* taking into account the (identically fulfilled) Bianchi identities. The relation of arbitrary functions in the seminal paper of Howe to supergravity multiplets is clarified. The local supersymmetry transformations remain the same, but, somewhat surprisingly, the transformations of zweibein and Rarita-Schwinger field decouple from those of the superconnection multiplet. A method emerges naturally, how to construct 'non-Einsteinian' supergravity theories with nontrivial curvature and torsion in $d = 2$ which, apart from their intrinsic interest, may be relevant for models of super black holes and for novel generalizations in superstring theories. Several explicit examples of such models are presented, some of which immediately allow a dilatonic formulation for the bosonic part of the action.

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1 Introduction

Despite the fact that no tangible direct evidence for supersymmetry so far has been discovered in Nature, supersymmetry since its discovery [1] managed to retain continual interest: within the aim to arrive at a fundamental 'theory of everything' first in supergravity [2] in $d = 4$, then in generalizations to higher dimensions of higher N [3], and finally incorporated as a low energy limit of superstrings [4] or, recently, of even more fundamental theories [5] in 11 dimensions.

Even before the advent of strings and superstrings the importance of studies in $1 + 1$ 'space-time' had been emphasized [6] in connection with the study of possible superspace formulations [7]. To the best of our knowledge to this day, however, no attempt has been made to generalize the supergravity formulation of (trivial) Einstein-gravity in $d = 2$ to the consideration of two-dimensional $(1, 1)$ supermanifolds for which the condition of vanishing (bosonic) torsion is removed. Only attempts to formulate theories with higher power of curvature (at vanishing torsion) seem to exist [8]. There seem to be only very few exact solutions of supergravity in $d = 4$ as well [9].

Especially at times, when the number of arguments in favour of the existence of an, as yet undiscovered, fundamental theory increase [3] it may seem appropriate to also exploit — if possible — *all* (super-)geometrical generalizations of the two-dimensional stringy world sheet. That such an undertaking can be (and indeed is) successful is suggested by the recent much improved insight, achieved for all (non-supersymmetric) two-dimensional diffeomorphism invariant theories, including dilaton theory, and permitting besides curvature also torsion [10] in the most general manner [11, 12, 13]. In the absence of matter-fields (non-geometrical degrees of freedom) all these models are integrable at the classical level and admit the analysis of all global solutions [14, 12]. Integrability of two-dimensional gravity coupled to chiral fermions was demonstrated in [15, 16]. Even the general aspects of quantization of any such theory now seem to be well understood [17, 11, 18, 19]. By contrast in the presence of matter, and if black holes like singularities occur in such models, integrable solutions are known only for very few cases. These include interactions with fermions of one chirality [15], and if scalar fields are present, only the dilaton black hole [20] and models which have asymptotical Rindler behaviour [21]. Therefore, a supersymmetric extension of the matterless case suggests that the solvability may carry over, in general. Then 'matter' could be represented by superpartners of the geometric bosonic field variables.

A straightforward approach would consist in a repetition of the calculation of [6] with nonvanishing bosonic torsion. However, already for the simpler case treated in that reference, the computational problems are considerable. We found it by far more suitable to solve the constraints *first*. Using the *inverse* supervierbein and superconnection — with conventional gauge-fixing — this task reduces to an algebraic (albeit still lengthy) problem. By construction this solution ful-

fils the Bianchi identities, but the latter are used nevertheless to determine the components of supertorsion and supercurvature.

Section 2 is devoted to the general definitions of superspace used in our present paper. In section 3 after introducing the torsion constraints *without* the requirement of vanishing bosonic torsion we fix the gauge in a way which later will turn out to be the correct one so that the remaining supergravity transformations are indeed the local generalizations of rigid supersymmetry. The constraints are solved in section 4 yielding the supermultiplets of vierbein and superconnection in terms of an arbitrary supergravity and a superconnection (or supertorsion) multiplet. In section 5 we compute the torsion and curvature components of superspace. The residual symmetry transformations of the supermultiplets of vierbein and Lorentz connection are contained in section 6, whereas finally in section 7 several simple examples of general (non-Einsteinian) supergravity actions within the superspace approach are constructed. In two appendices we describe notations and conventions.

2 Geometry of superspace

In $d = 2$ we consider a superspace with two commuting (bosonic) and two anticommuting (Grassmann or spinor) coordinates $z^M = \{x^m, \theta^\mu\}$ where lower case Latin ($m = 0, 1$) and Greek indices ($\mu = 1, 2$) denote commuting and anticommuting coordinates, respectively:

$$z^M z^N = z^N z^M (-1)^{MN}. \tag{1}$$

Within our conventions for Majorana spinors (cf. appendix A) the first anticommuting element of the Grassmann algebra is supposed to be real, $\theta^{1*} = \theta^1$, while the second one is purely imaginary, $\theta^{2*} = -\theta^2$.

Our construction is based on differential geometry of superspace. We shall not deal with subtle mathematical definitions [22]. For our purpose it is sufficient to follow one simple working rule allowing to generalize ordinary formulas of differential geometry to superspace.

The exterior derivative operator in superspace

$$d = dz^M \partial_M \tag{2}$$

is invariant under arbitrary nondegenerate coordinate changes $z^M \to z'^M(z)$:

$$dz^M \partial_M = dz'^M \frac{\partial z^L}{\partial z'^M} \frac{\partial z'^N}{\partial z^L} \partial_{N'} \tag{3}$$

Summation over repeated indices is assumed, and derivatives are always supposed to be the left derivatives. From (3) follows our simple basic rule: Any formula of differential geometry in ordinary space can be taken over to superspace if the

summation is always performed from the upper left corner to the lower right one with no indices in between ('ten to four'), and the order of the indices in each term of the expression must be the same. Otherwise an appropriate factor (-1) must be included. E. g. the invariant interval reads

$$ds^2 = dz^M \otimes dz^N G_{NM} = dz^M \otimes dz^N G_{MN} (-1)^{MN}, \tag{4}$$

where G_{MN} is the superspace metric. This metric can be used to lower indices of a vector field,

$$V_M = V^N G_{NM} = G_{MN} V^N (-1)^N. \tag{5}$$

The generalization to an arbitrary tensor is obvious. Defining the inverse metric according to the rule

$$V^M = G^{MN} V_N = V_N G^{NM} (-1)^N, \tag{6}$$

and demanding that sequential lowering and raising indices to be the identical operation yields the main property of the inverse metric

$$G^{MN} G_{NP} = \delta_P{}^M (-1)^{MP} = \delta_P{}^M (-1)^M = \delta_P{}^M (-1)^P. \tag{7}$$

The last identities follow from the diagonality of the Kronecker symbol $\delta_P{}^M = \delta_P^M$. Thus the inverse metric is not an inverse matrix in the usual sense. From (5) the quantity

$$V^2 = V^M V_M = V_M V^M (-1)^M \tag{8}$$

is a scalar, (but e. g. $V_M V^M$ is not!).

We assume that our superspace is equipped with a Riemann-Cartan geometry that is with a metric and with a metrical connection $\Gamma_{MN}{}^P$. The latter defines the covariant derivative of a tensor field. Covariant derivatives of a vector V^N and covector V_N read as

$$\nabla_M V^N = \partial_M V^N + V^P \Gamma_{MP}{}^N (-1)^{PM}, \tag{9}$$

$$\nabla_M V_N = \partial_M V_N - \Gamma_{MN}{}^P V_P. \tag{10}$$

The metricity condition for the metric is

$$\nabla_M G_{NP} = \partial_M G_{NP} - \Gamma_{MN}{}^R G_{RP} - \Gamma_{MP}{}^R G_{RN} (-1)^{NP} = 0. \tag{11}$$

The action of an (anti)commutator of covariant derivatives,

$$[\nabla_M, \nabla_N\} = \nabla_M \nabla_N - \nabla_N \nabla_M (-1)^{MN} \tag{12}$$

on a vector field (5),

$$[\nabla_M, \nabla_N\} V_P = -R_{MNP}{}^R V_R - T_{MN}{}^R \nabla_R V_P, \tag{13}$$

is defined in terms of curvature and torsion:

$$R_{MNP}{}^{R} = \partial_M \Gamma_{NP}{}^{R} - \Gamma_{MP}{}^{S}\Gamma_{NS}{}^{R}(-1)^{N(S+P)} - (M \leftrightarrow N)(-1)^{MN}, \tag{14}$$
$$T_{MN}{}^{R} = \Gamma_{MN}{}^{R} - \Gamma_{NM}{}^{R}(-1)^{MN} \tag{15}$$

In our construction we use Cartan variables: the superspace vierbein $E_M{}^A$ and the superconnection $\Omega_{MA}{}^{B}$. Capital Latin indices from the beginning of the alphabet ($A = a, \alpha$) transform under the Lorentz group as a vector ($a = 0, 1$) and spinor ($\alpha = 1, 2$), respectively. Cartan variables are defined by

$$G_{MN} = E_M{}^A E_N{}^B \eta_{BA}(-1)^{AN}, \tag{16}$$

and the metricity condition

$$\nabla_M E_N{}^A = \partial_M E_N{}^A - \Gamma_{MN}{}^P E_P{}^A + E_N{}^B \Omega_{MB}{}^A (-1)^{M(B+N)} = 0. \tag{17}$$

Raising and lowering of the anholonomic indices $(A, B, \dots)$ is performed by the superspace Minkowski metric

$$\eta_{AB} = \begin{pmatrix} \eta_{ab} & 0 \\ 0 & \epsilon_{\alpha\beta} \end{pmatrix}, \quad \eta^{AB} = \begin{pmatrix} \eta^{ab} & 0 \\ 0 & \epsilon^{\alpha\beta} \end{pmatrix}, \tag{18}$$

consisting of the two-dimensional Minkowskian metric $\eta_{ab} = \eta^{ab} = \mathrm{diag}(+-)$ and $\epsilon_{\alpha\beta}$, the totally antisymmetric (Levi-Civita) tensor defined in appendix A. The Minkowski metric and its inverse (18) in superspace obey

$$\eta_{AB} = \eta_{BA}(-1)^A, \quad \eta^{AB}\eta_{BC} = \delta_C{}^A(-1)^A. \tag{19}$$

The transformation of anholonomic indices $(A, B, \dots)$ into holonomic indices $(M, N, \dots)$ and vice versa is performed using the supervierbein and its inverse $E_A{}^M$ defined as

$$E_A{}^M E_M{}^B = \delta_A{}^B, \quad E_M{}^A E_A{}^N = \delta_M{}^N. \tag{20}$$

The metric (18) is invariant under the Lorentz group acting on tensor indices from the beginning of the alphabet. In fact (18) is not unique in this respect because $\epsilon_{\alpha\beta}$ may be multiplied by an arbitrary nonzero factor. This may represent a freedom to generalize our present approach. In fact, in order to have a correct dimension of all terms in the line element of superspace, that factor should carry the dimension of length. A specific choice for it presents a freedom in approaches to supersymmetry. In the following this factor will be suppressed. Therefore any apparent differences in dimensions between terms below are not relevant.

The metricity condition (17) formally establishes a one-to-one correspondence between the metrical connection $\Gamma_{MN}{}^P$ and the superconnection $\Omega_{MA}{}^B$. Together with (11) it implies

$$\nabla_M \eta_{AB} = 0 \tag{21}$$

and the symmetry property

$$\Omega_{MAB} + \Omega_{MBA}(-1)^{AB} = 0. \tag{22}$$

In general, the superconnection $\Omega_{MA}{}^B$ is not related to Lorentz transformations alone. The Lorentz connection in superspace, as will be seen from section 3, must have a specific form and is defined (in $d = 2$) by Ω_M, a superfield with one vector index,

$$\Omega_{MA}{}^B = \Omega_M L_A{}^B, \tag{23}$$

where

$$L_A{}^B = \begin{pmatrix} \epsilon_a{}^b & 0 \\ 0 & -\frac{1}{2}\gamma^5{}_\alpha{}^\beta \end{pmatrix} \tag{24}$$

contains the Lorentz generators in the bosonic and fermionic sectors. Here the factor in front of γ^5 is fixed by the requirement that under Lorentz transformations γ-matrices are invariant under simultaneous rotations of vector and spinor indices. Definition and properties of γ-matrices are given in appendix A. $L_A{}^B$ has the properties

$$L_{AB} = -L_{BA}(-1)^A, \quad L_A{}^B L_B{}^C = \begin{pmatrix} \delta_a{}^c & 0 \\ 0 & \frac{1}{4}\delta_\alpha{}^\gamma \end{pmatrix}, \quad \nabla_M L_A{}^B = 0. \tag{25}$$

The superconnection $\Omega_{MA}{}^B$ in the form (23) is very restricted, because the original 32 independent superfield components for the Lorentz superconnection reduce to 4. As a consequence, (17) with (23) also entails restrictions on the metric connection $\Gamma_{MN}{}^P$.

In terms of the connection (23) covariant derivatives of a Lorentz supervector read

$$\nabla_M V^A = \partial_M V^A + \Omega_M V^B L_B{}^A, \tag{26}$$

$$\nabla_M V_A = \partial_M V_A - \Omega_M L_A{}^B V_B. \tag{27}$$

The (anti)commutator of covariant derivatives,

$$[\nabla_M, \nabla_N\} V_A = -R_{MNA}{}^B V_B - T_{MN}{}^P \nabla_P V_A, \tag{28}$$

is defined by the same expressions for curvature and torsion as given by (14) and (15), which in Cartan variables become

$$R_{MNA}{}^B = \partial_M \Omega_{NA}{}^B - \Omega_{MA}{}^C \Omega_{NC}{}^B (-1)^{N(A+C)} - (M \leftrightarrow N)(-1)^{MN}, \tag{29}$$

$$T_{MN}{}^A = \partial_M E_N{}^A + E_N{}^B \Omega_{MB}{}^A (-1)^{M(B+N)} - (M \leftrightarrow N)(-1)^{MN}. \tag{30}$$

In terms of the Lorentz connection (23) the curvature does not contain quadratic terms

$$R_{MNA}{}^{B} = \left(\partial_M \Omega_N - \partial_N \Omega_M (-1)^{MN}\right) L_A{}^{B} = F_{MN} L_A{}^{B}. \tag{31}$$

In the calculations below we have found it extremely convenient to work directly in the anholonomic basis

$$D_A = E_A{}^{M} \partial_M,$$

defined by the inverse supervierbein. In this basis curvature (31) and torsion (30) become

$$F_{AB} = -C_{AB}{}^{C} \Omega_C + D_A \Omega_B - D_B \Omega_A (-1)^{AB}, \tag{32}$$

$$T_{AB}{}^{C} = -C_{AB}{}^{C} + \Omega_A L_B{}^{C} - \Omega_B L_A{}^{C} (-1)^{AB}, \tag{33}$$

where

$$C_{AB}{}^{C} = \left(E_A{}^{N} \partial_N E_B{}^{M} - E_B{}^{N} \partial_N E_A{}^{M} (-1)^{AB}\right) E_M{}^{C}$$

are the coefficients of anholonomicity. Beside the inverse supervierbein $E_A{}^{M}$ also the use of the Lorentz superconnection in the anholonomic basis, $\Omega_A = E_A{}^{M} \Omega_M$ has proved to be crucial for our approach.

Ricci tensor and scalar curvature of the manifold are

$$R_{AB} = R_{ACB}{}^{C} (-1)^{C(B+C)} = -L_B{}^{C} F_{CA} (-1)^{AB}, \tag{34}$$

$$R = R_A{}^{A} (-1)^{A} = -L^{AB} F_{BA}. \tag{35}$$

Curvature and torsion satisfy the Bianchi identities

$$\begin{aligned} &\nabla_A R_{BCDE} + \nabla_B R_{CADE} (-1)^{A(B+C)} + \nabla_C R_{ABDE} (-1)^{C(A+B)} \\ &+ T_{AB}{}^{F} R_{FCDE} + T_{BC}{}^{F} R_{FADE} (-1)^{A(B+C)} + T_{CA}{}^{F} R_{FBDE} (-1)^{C(A+B)} = 0 \end{aligned} \tag{36}$$

and

$$\begin{aligned} &\nabla_A T_{BCD} + \nabla_B T_{CAD} (-1)^{A(B+C)} + \nabla_C T_{ABD} (-1)^{C(A+B)} = \\ &\qquad R_{ABCD} + R_{BCAD} (-1)^{A(B+C)} + R_{CABD} (-1)^{C(A+B)} \\ &\qquad - T_{AB}{}^{E} T_{ECD} - T_{BC}{}^{E} T_{EAD} (-1)^{A(B+C)} - T_{CA}{}^{E} T_{EBD} (-1)^{C(A+B)} \end{aligned} \tag{37}$$

where with $R_{ABC}{}^{D} = F_{AB} L_C{}^{D}$ the Bianchi identity (36) simplifies considerably

$$\begin{aligned} &\nabla_A F_{BC} + \nabla_B F_{CA} (-1)^{A(B+C)} + \nabla_C F_{AB} (-1)^{C(A+B)} \\ &\qquad + T_{AB}{}^{D} F_{DC} + T_{BC}{}^{D} F_{DA} (-1)^{A(B+C)} + T_{CA}{}^{D} F_{DB} (-1)^{C(A+B)} = 0. \end{aligned} \tag{38}$$

3 Supergravity Constraints and Gauge Fixing

Generally superdiffeomorphism invariant theories may be constructed easily from supervierbein and superconnection. However their flat limit does not show rigid supersymmetry. What is commonly defined to be 'supergravity' is the restricted case which is invariant under diffeomorphisms in the bosonic subspace, local Lorentz boosts, and local supersymmetry transformations mixing bosonic and fermionic fields, all parameters depending only on the *bosonic* coordinates (x^m). As a consequence of the uniqueness theorem of suitably redefined supersymmetry transformations in rigid superspace [23] it is natural to postulate

$$T_{\alpha\beta}{}^a = 2i(\gamma^a\epsilon)_{\alpha\beta}, \quad T_{\alpha\beta}{}^\gamma = 0. \tag{39}$$

These constraints are identical to the first two standard constraints, also imposed by Howe [6]. We find, though, that the third constraint $T_{ab}{}^c = 0$ [6] specifying the bosonic part of the torsion to be 'Einsteinian' is not mandatory for supersymmetry transformations and may be dropped. This leads to a special type of Riemann-Cartan superspace, where we retain the 'maximal amount' of (1,1)-superspace structure in the tangent space. In particular, (39) should be invariant under superdiffeomorphism and Lorentz transformation. The first requirement is trivially fulfilled. The second one produces precisely the restriction on $\Omega_{MA}{}^B$ anticipated in section 2. The superpartner of an inverse zweibein $e_a{}^m$ is supposed to be a Rarita-Schwinger field $\chi_a{}^\mu$ carrying one vector and one spinor index. Within the superspace approach to supergravity diffeomorphisms of the bosonic sector and supersymmetry transformations must appear as subgroups of the full superspace diffeomorphisms

$$z'^M = w^M(z) = w^{(0)M} + \theta^\nu w^{(1)}{}_\nu{}^M + \frac{1}{2}\theta\bar{\theta}w^{(2)M}, \tag{40}$$

where $w^{(0,1,2)}$ are functions of x^m only. Since all symmetry parameters of a proper supergravity model are assumed to depend on x^m alone it is natural to use the functions $w^{(1)}$ and $w^{(2)}$ to fix certain components of the supervierbein. Under the transformation (40) the components $E_\alpha{}^M$ transform as

$$\begin{aligned} E'_\alpha{}^m &= E_\alpha{}^n\partial_n w^m + E_\alpha{}^\nu\left(w^{(1)}{}_\nu{}^m + \theta_\nu w^{(2)m}\right), \\ E'_\alpha{}^\mu &= E_\alpha{}^n\partial_n w^\mu + E_\alpha{}^\nu\left(w^{(1)}{}_\nu{}^\mu + \theta_\nu w^{(2)\mu}\right). \end{aligned} \tag{41}$$

Expanding the inverse supervierbein in θ

$$E_A{}^M = E^{(0)}{}_A{}^M + \theta^\nu E^{(1)}{}_{\nu A}{}^M + \frac{1}{2}\theta\bar{\theta}E^{(2)}{}_A{}^M,$$

one easily sees that if $\det E^{(0)}{}_\alpha{}^\mu \neq 0$ then one may always use the functions $w^{(1)}{}_\nu{}^M$ to fix $E^{(0)}{}_\alpha{}^M$:

$$E^{(0)}{}_\alpha{}^m = 0, \quad E^{(0)}{}_\alpha{}^\mu = \delta_\alpha{}^\mu. \tag{42}$$

In the next step the functions $w^{(2)m}$ and $w^{(2)\mu}$ may be used to get rid of the antisymmetric parts of the first order components

$$E^{(1)}{}_{\nu\alpha}{}^{m} = E^{(1)}{}_{\alpha\nu}{}^{m}, \quad E^{(1)}{}_{\nu\alpha}{}^{\mu} = E^{(1)}{}_{\alpha\nu}{}^{\mu}. \tag{43}$$

Under the Lorentz boost the superconnection transforms according to

$$\Omega'_A = S^{-1}{}_A{}^B(\Omega_B - D_B W), \tag{44}$$

where $S^{-1}{}_A{}^B$ is the inverse Lorentz transformation matrix corresponding to the boost parameter

$$W = W^{(0)} + \theta^\nu W^{(1)}{}_\nu + \frac{1}{2}\theta\bar{\theta}W^{(2)}, \tag{45}$$

which also represents a superfield. The freedom in choosing $W^{(1)}$ and $W^{(2)}$ allows us to set

$$\Omega^{(0)}{}_\alpha = 0, \quad \Omega^{(1)}{}_{\nu\alpha} = \Omega^{(1)}{}_{\alpha\nu}, \tag{46}$$

respectively. This exhausts all group parameters to first and second order in θ. So the remaining free parameters of the symmetry transformations are $w^{(0)m}$, $w^{(0)\mu}$, and $W^{(0)}$ which are functions of x^m alone and (as we shall see later) indeed describe bosonic diffeomorphisms, local supersymmetry transformations, and local Lorentz boosts, respectively. The expansion in θ of the supervierbein and the Lorentz superconnection, including the gauge fixing (42), (43) and (46), may be written as

$$E_a{}^m = e_a{}^m + \theta^\nu f_{\nu a}{}^m + \frac{1}{2}\theta\bar{\theta}g_a{}^m, \tag{47}$$

$$E_a{}^\mu = \chi_a{}^\mu + \theta^\nu f_{\nu a}{}^\mu + \frac{1}{2}\theta\bar{\theta}g_a{}^\mu, \tag{48}$$

$$E_\alpha{}^m = \theta^\nu f_{\nu\alpha}{}^m + \frac{1}{2}\theta\bar{\theta}g_\alpha{}^m, \qquad f_{\nu\alpha}{}^m = f_{\alpha\nu}{}^m, \tag{49}$$

$$E_\alpha{}^\mu = \delta_\alpha{}^\mu + \theta^\nu f_{\nu\alpha}{}^\mu + \frac{1}{2}\theta\bar{\theta}g_\alpha{}^\mu, \qquad f_{\nu\alpha}{}^\mu = f_{\alpha\nu}{}^\mu, \tag{50}$$

$$\Omega_a = \omega_a + \theta^\nu \bar{u}_{\nu a} + \frac{1}{2}\theta\bar{\theta}\upsilon_a, \tag{51}$$

$$\Omega_\alpha = \theta^\nu \rho_{\nu\alpha} + \frac{1}{2}\theta\bar{\theta}\upsilon_\alpha, \qquad \rho_{\nu\alpha} = \rho_{\alpha\nu}. \tag{52}$$

Here $e_a{}^m$, $\chi_a{}^\mu$, and ω_a are the inverse zweibein, the Rarita-Schwinger field and the (bosonic) Lorentz connection, respectively. Part of the other components will be found in the next section as the solution to the torsion constraints while the rest are needed to form supermultiplets for e, χ, and ω.

The Wess-Zumino type gauge (42), (43) and (46) is similar to that used by Howe [6] but it is applied to the *inverse* supervierbein. After solution of the

constraints and inverting the supervierbein we shall get a result consistent with his gauge and vice versa. So the gauges are equivalent. In fact, what amounts to 'fixing the gauge' is similar to what happens in the coset approach to 'gauging' supersymmetry into supergravity [24].

4 Solution of the constraints

Writing the constraints (39) in equivalent form

$$T_{\alpha\beta}{}^M = 2i(\gamma^a\epsilon)_{\alpha\beta}E_a{}^M, \tag{53}$$

or, after contracting with suitable γ-matrices (cf. appendix A),

$$T_{\alpha\beta}{}^M(\epsilon\gamma^5)^{\beta\alpha} = 0, \tag{54}$$

$$T_{\alpha\beta}{}^M(\epsilon\gamma_a)^{\beta\alpha} = -4iE_a{}^M, \tag{55}$$

one observes that (55) simply becomes the relation expressing $E_a{}^M$ in terms of what has been obtained by the solution of (54). Thus only the latter relation must be solved, which contains through $T_{\alpha\beta}{}^M$ the components $E_\alpha{}^M$ of the inverse supervierbein and Ω_α of the Lorentz superconnection. As we shall see all components of (47)–(50) and (52) can be solved in terms of the supermultiplet $e_a{}^m$, $\chi_a{}^\mu$ and an arbitrary scalar field A without using the Bianchi identities at this point. If instead the supervierbein $E_M{}^A$ is considered as an independent variable then the constraints (39) either both contain the vierbein and its inverse thus providing a major computational problem, or, when written for $T_{MN}{}^A$ require the knowledge of other torsion components. In that last case, which had been exploited in [6], the prior solution of the Bianchi identities is inevitable.

To solve equations (54), (55) we decompose ρ and f in (47)–(52) in the basis for symmetric matrices:

$$\begin{aligned} \rho_{\nu\alpha} &= A(\gamma^5\epsilon)_{\nu\alpha} + i\rho_a(\gamma^a\epsilon)_{\nu\alpha}, \\ f_{\nu\alpha}{}^m &= f^m(\gamma^5\epsilon)_{\nu\alpha} + f_a{}^m(\gamma^a\epsilon)_{\nu\alpha}, \\ f_{\nu\alpha}{}^\mu &= f^\mu(\gamma^5\epsilon)_{\nu\alpha} + f_a{}^\mu(\gamma^a\epsilon)_{\nu\alpha}. \end{aligned} \tag{56}$$

In addition we separate (cf. appendix A) the vector-spinor according to

$$\chi^a = \chi\gamma^a + \lambda^a, \quad \gamma^a\lambda_a = 0 \tag{57}$$

into a Rarita-Schwinger field λ^a and a spinor χ. Then to zeroth order in θ the equations (54) and (55) yield

$$f^m = 0, \quad f_a{}^m = -ie_a{}^m, \tag{58}$$

$$f^\mu = 0, \quad f_a{}^\mu = -i\chi_a{}^\mu. \tag{59}$$

To first order in θ one obtains

$$g_\alpha{}^m = 2\bar{\lambda}^m{}_\alpha, \quad f_{\nu a}{}^m = i(\gamma^m \bar{\chi}_a)_\nu, \tag{60}$$

and

$$g_\alpha{}^\mu = 2\bar{\lambda}^b{}_\alpha \chi_b{}^\mu - \frac{1}{2} A \delta_\alpha{}^\mu + \frac{i}{2} \rho_b (\gamma^b \gamma^5)_\alpha{}^\mu, \tag{61}$$

$$f_{\nu a}{}^\mu = i(\gamma^b \bar{\chi}_a)_\nu \chi_b{}^\mu - \frac{i}{2} A \gamma_{a\nu}{}^\mu - \frac{1}{2} \rho_a \gamma^5{}_\nu{}^\mu. \tag{62}$$

The transformation of the lower case Latin indices from the beginning of the alphabet to the ones from the middle is performed by using the zweibein, e. g. $\chi_m{}^\alpha = e_m{}^a \chi_a{}^\alpha$, $\gamma_m = e_m{}^a \gamma_a$. The two-dimensional bosonic metric $g_{mn} = e_m{}^a e_{na}$ is used for raising and lowering the indices.

Eqs. (54), (55) in second order of θ determine two functions in the Lorentz superconnection

$$\rho_a = -\epsilon_a{}^b c_b - 4i(\lambda_a \gamma^5 \bar{\chi}), \tag{63}$$

$$\upsilon_\alpha = 4\epsilon^{mn} (\widetilde{\nabla}_m \bar{\chi}_n)_\alpha + 2c_b (\gamma^5 \bar{\lambda}^b)_\alpha - 2iA(\gamma^5 \bar{\chi})_\alpha \tag{64}$$

where the trace of the anholonomicity coefficients c_b and the covariant derivative $\widetilde{\nabla}_m$ are defined in appendix B.

The remaining components of the inverse supervierbein (47)–(50) are also obtained from that order as

$$g_a{}^m = -2(\lambda^m \bar{\chi}_a), \tag{65}$$

$$g_a{}^\mu = i\epsilon^{mn} (\widetilde{\nabla}_m \chi_n \gamma_a \gamma^5)^\mu + \frac{i}{2} c_b (\chi_a \gamma^b)^\mu + A(\chi \gamma_a)^\mu. \tag{66}$$

Thus after some rearrangements the general solution to the constraints (39) in the gauge (42), (43) and (46) can be summarized as ($\lambda^2 = \lambda^a \bar{\lambda}_a$)

$$E_a{}^m = e_a{}^m + i(\theta \gamma^m \bar{\chi}_a) - \theta\bar{\theta}(\lambda^m \bar{\chi}_a), \tag{67}$$

$$\begin{aligned} E_a{}^\mu = {} & \chi_a{}^\mu + i(\theta \gamma^b \bar{\chi}_a) \chi_b{}^\mu - \frac{i}{2} A(\theta \gamma_a)^\mu - \frac{1}{2} \rho_a (\theta \gamma^5)^\mu \\ & + \frac{1}{2} \theta\bar{\theta} \left[i\epsilon^{mn} (\widetilde{\nabla}_m \chi_n \gamma_a \gamma^5)^\mu + \frac{i}{2} c_b (\chi_a \gamma^b)^\mu + A(\chi \gamma_a)^\mu \right], \end{aligned} \tag{68}$$

$$E_\alpha{}^m = i(\gamma^m \bar{\theta})_\alpha + \theta\bar{\theta} \bar{\lambda}^m{}_\alpha, \tag{69}$$

$$E_\alpha{}^\mu = \delta_\alpha{}^\mu + i(\gamma^b \bar{\theta})_\alpha \chi_b{}^\mu - \frac{1}{2} \theta\bar{\theta} \left[\lambda^2 \delta_\alpha{}^\mu + \frac{1}{2} A \delta_\alpha{}^\mu + \frac{i}{2} c_b \gamma^b{}_\alpha{}^\mu \right], \tag{70}$$

for the inverse supervierbein and

$$\Omega_a = \omega_a + (\theta \bar{u}_a) + \frac{1}{2} \theta\bar{\theta} \upsilon_a \tag{71}$$

$$\begin{aligned} \Omega_\alpha = {} & A(\theta \gamma^5 \epsilon)_\alpha + i\rho_b (\theta \gamma^b \epsilon)_\alpha \\ & + \frac{1}{2} \theta\bar{\theta} \left[4\epsilon^{mn} (\widetilde{\nabla}_m \bar{\chi}_n)_\alpha + 2c_b (\gamma^5 \bar{\lambda}^b)_\alpha - 2iA(\gamma^5 \bar{\chi})_\alpha \right], \end{aligned} \tag{72}$$

for the superconnection where ρ_a by (63) is a function of $e_a{}^m$ and $\chi_a{}^\mu$. The very existence of this solution proves that the torsion constraints are indeed consistent with the Bianchi identities, without having to solve the latter at all. Inspecting (67)–(72) shows that after solution of the constraints we are left with two 'supermultiplets': the supergravity multiplet $\mathcal{E} = \{e_a{}^m, \chi_a{}^\mu, A\}$ and the Lorentz connection supermultiplet $\Omega_a = \{\omega_a, u_a{}^\nu, \upsilon_a\}$. The supergravity multiplet consists of the inverse zweibein $e_a{}^m$, the vector-spinor field $\chi_a{}^\mu$, and the scalar field A. The components of $\mathcal{E}$ originate from components of different superfields whereas the Lorentz connection supermultiplet represents one superfield Ω_a. It does not enter the constraint equation at all and so remains completely arbitrary. In the next section we shall see that the parameters of local symmetry transformations depend only on $\mathcal{E}$.

Comparing our result with the one by Howe [6] we observe that our function A (up to a factor) is identical to A of that paper. There it appeared as the first component of a scalar superfield entering a general solution of the Bianchi identities. Our use of the inverse supervierbein as a primary field thus not only allows to postpone the discussion of the Bianchi identities (thus simplifying the calculations), but also clarifies the geometrical meaning of A as the scalar component of $\mathcal{E}$.

5 Superspace curvature and torsion

In order to compute superspace curvature and torsion in terms of the supermultiplets $\mathcal{E}$ and Ω one needs the explicit form of the supervierbein $E_M{}^A$. Solving one of the eqs. (20) we find the components

$$E_m{}^a = e_m{}^a - 2i(\theta\gamma^a\bar{\chi}_m) + \frac{1}{2}\theta\bar{\theta}Ae_m{}^a, \tag{73}$$

$$\begin{aligned} E_m{}^\alpha = & -\chi_m{}^\alpha + \frac{i}{2}A(\theta\gamma_m)^\alpha + \frac{1}{2}\rho_m(\theta\gamma^5)^\alpha \\ & + \frac{1}{2}\theta\bar{\theta}\left[-i\epsilon^{np}(\widetilde{\nabla}_n\chi_p\gamma_m\gamma^5)^\alpha + \lambda^2(\chi\gamma_m)^\alpha + \frac{1}{2}A(\chi\gamma_m)^\alpha + \frac{3}{2}A\lambda_m{}^\alpha\right], \end{aligned} \tag{74}$$

$$E_\mu{}^a = i(\theta\gamma^a\epsilon)_\mu, \tag{75}$$

$$E_\mu{}^\alpha = \delta_\mu{}^\alpha\left(1 - \frac{1}{4}\theta\bar{\theta}A\right). \tag{76}$$

Eqs. (73)–(76) in terms of components indeed coincide (up to misprints) with those of Howe [6].

Now we are able to compute the anholonomic components of curvature and torsion defined in section 2. Only at this point it becomes useful to take the Bianchi identities into account. For the constraints (39) a straightforward, but

tedious calculation yields expressions for torsion

$$T_{\alpha\beta}{}^{\gamma} = 0, \tag{77}$$
$$T_{\alpha\beta}{}^{a} = 2i(\gamma^a\epsilon)_{\alpha\beta}, \tag{78}$$
$$T_{\alpha a}{}^{\beta} = \frac{i}{2}S\gamma_{a\alpha}{}^{\beta} + \frac{1}{2}\epsilon_a{}^b T_b \gamma^5{}_\alpha{}^\beta, \tag{79}$$
$$T_{\alpha a}{}^{b} = -T_{a\alpha}{}^{b} = 0, \tag{80}$$
$$T_{ab}{}^{\alpha} = \frac{1}{2}\epsilon_{ab}(\epsilon\gamma^5)^{\alpha\beta}\nabla_\beta S, \tag{81}$$
$$T_{ab}{}^{c} = \delta_a{}^c T_b - \delta_b{}^c T_a \tag{82}$$

and curvature

$$F_{\alpha\beta} = 2S(\gamma^5\epsilon)_{\alpha\beta} + 2iT_a(\gamma^a\gamma^5\epsilon)_{\alpha\beta}, \tag{83}$$
$$F_{\alpha a} = -F_{a\alpha} = i(\gamma_a\gamma^5)_\alpha{}^\beta\nabla_\beta S + \epsilon_a{}^b\nabla_\alpha T_b, \tag{84}$$
$$F_{ab} = \epsilon_{ab}\left[-\frac{1}{2}\epsilon^{\alpha\beta}\nabla_\beta\nabla_\alpha S + S^2 - \nabla_c T^c + T_c T^c\right] \tag{85}$$

in terms of S and T_a, a scalar and a vector superfield. In order to find out how they depend on the components of $E_M{}^A$, eqs. (73)–(76), it suffices to compute the l. h. s. of (79) or (83) and to compare both sides of the equation. The scalar superfield turns out to depend on $\mathcal{E}$ alone

$$\begin{aligned} S = {} & A + 2\epsilon^{mn}(\theta\gamma^5\widetilde{\nabla}_m\bar{\chi}_n) + 2i(\theta\bar{\chi})\lambda^2 - 2iA(\theta\bar{\chi}) \\ & - \frac{1}{2}\theta\bar{\theta}\left[\frac{1}{2}\widetilde{R} - 4i\epsilon^{mn}(\chi\gamma^5\widetilde{\nabla}_m\bar{\chi}_n) + 4i\widetilde{\nabla}_a(\lambda^a\bar{\chi}) + 4\chi^2\lambda^2 + A(\chi^a\bar{\chi}_a) + A^2\right] \end{aligned} \tag{86}$$

where $\widetilde{R}$ is the curvature scalar for vanishing (bosonic) torsion (cf. appendix B). The vector superfield is nothing else than the trace of the supertorsion

$$T_B = T_{AB}{}^A(-1)^{A+AB} = (T_b, T_\beta = 0)\,. \tag{87}$$

The components of $T_a = -C_a + \epsilon_a{}^b\Omega_b$, where $C_a = C_{ba}{}^b$,

$$T_a = t_a + (\theta\bar{\tau}_a) + \frac{1}{2}\theta\bar{\theta}s_a, \tag{88}$$

depend on both $\mathcal{E}$ and Ω,

$$t_a = \hat{t}_a - 4i(\lambda_a\bar{\chi}) \tag{89}$$
$$\begin{aligned} (\theta\bar{\tau}_a) = {} & 2i\epsilon^{mn}(\theta\gamma_a\gamma^5\widetilde{\nabla}_m\bar{\chi}_n) - i\epsilon_a{}^b c_c(\theta\gamma^c\gamma^5\bar{\chi}_b) \\ & + 4\chi^2(\theta\bar{\lambda}_a) - A(\theta\bar{\chi}_a) + \epsilon_a{}^b(\theta\bar{u}_b), \end{aligned} \tag{90}$$
$$\begin{aligned} s_a = {} & -\partial_a A + 4\epsilon^{mn}(\widetilde{\nabla}_m\chi_n\gamma_a\gamma^5\bar{\chi}) - 2\epsilon_a{}^b c_c(\lambda^c\gamma^5\bar{\chi}_b) \\ & + 2iA(\chi\bar{\lambda}_a) + \epsilon_a{}^b\upsilon_b. \end{aligned} \tag{91}$$

Here $\hat{t}_a$ is the trace of the torsion of the two-dimensional bosonic subspace (cf. appendix B). The expressions for supertorsion and supercurvature obtained here will be used in section 7 for the construction of generalized supergravity Lagrangians.

6 Symmetry transformations

Since the gauge was not fixed completely several parameters are still free: the zeroth components of superspace diffeomorphisms $w^{(0)M}$ (cf. (40)) and the zeroth component of the Lorentz rotation $W^{(0)}$ in (45). They determine the remaining symmetry transformations. If our gauge fixing in section 3 was a suitable one we should obtain in this way the correct local transformations of supergravity.

Under infinitesimal superdiffeomorphisms parametrized by a vector superfield $\xi^M(z)$, and by an infinitesimal Lorentz (super-)boost with parameter $W(z)$ the inverse supervierbein and the anholonomic components of the Lorentz superconnection obey the transformation formulas

$$\begin{aligned}\delta E_A{}^M &= \xi^N \partial_N E_A{}^M - E_A{}^N \partial_N \xi^M - W L_A{}^B E_B{}^M,\\ \delta \Omega_A &= \xi^N \partial_N \Omega_A - W L_A{}^B \Omega_B - E_A{}^M \partial_M W.\end{aligned} \tag{92}$$

To find the explicit form of the remaining symmetry transformations after the gauge fixing of section 3 we decompose

$$\begin{aligned}\xi^m &= \zeta^m + \theta^\nu k_\nu{}^m + \frac{1}{2}\theta\bar{\theta} l^m,\\ \xi^\mu &= \zeta^\mu + \theta^\nu k_\nu{}^\mu + \frac{1}{2}\theta\bar{\theta} l^\mu,\\ W &= \omega + \theta^\nu k_\nu + \frac{1}{2}\theta\bar{\theta} l,\end{aligned} \tag{93}$$

where $\zeta^m(x)$, $\zeta^\mu(x)$ and $\omega(x)$ are the parameters of bosonic diffeomorphisms, supersymmetry transformations and Lorentz boosts, respectively. In section 3 we made use of the components to zero and first order in θ of (93) to argue that certain gauge conditions may be imposed. This, however, does not mean that those components are fixed (e. g. to zero), but only that the higher order components depend on the zero order ones and on the fields constituting the supermultiplets. In order to maintain the gauge conditions (42) and the first relation (46) we must have

$$\delta E^{(0)}{}_\alpha{}^m = 0, \quad \delta E^{(0)}{}_\alpha{}^\mu = 0, \quad \delta \Omega^{(0)}{}_\alpha = 0. \tag{94}$$

In terms of the transformation components in (93) one easily finds from (92)

$$\begin{aligned} k_\nu{}^m &= i(\gamma^m\bar\zeta)_\nu, \\ k_\nu{}^\mu &= i(\gamma^b\bar\zeta)_\nu\chi_b{}^\mu + \frac{1}{2}\omega\gamma^5{}_\nu{}^\mu, \\ k_\nu &= -A(\gamma^5\bar\zeta)_\nu - i\rho_b(\gamma^b\bar\zeta)_\nu. \end{aligned} \tag{95}$$

To be consistent with the remaining gauge conditions one has to solve the equations

$$\begin{aligned} \delta\left(E^{(1)}{}_{\nu\alpha}{}^m - E^{(1)}{}_{\alpha\nu}{}^m\right) &= 0, \\ \delta\left(E^{(1)}{}_{\nu\alpha}{}^\mu - E^{(1)}{}_{\alpha\nu}{}^\mu\right) &= 0, \\ \delta\left(\Omega^{(1)}{}_{\nu\alpha} - \Omega^{(1)}{}_{\alpha\nu}\right) &= 0 \end{aligned} \tag{96}$$

which determine the remaining functions l^m, l^μ, and l. To clarify the geometrical meaning of the transformations we write the final answer separately for bosonic diffeomorphisms and Lorentz rotations (setting $\zeta^\mu = 0$)

$$\xi^m = \zeta^m, \tag{97}$$

$$\xi^\mu = \frac{1}{2}\omega(\theta\gamma^5)^\mu, \tag{98}$$

$$W = \omega, \tag{99}$$

and for local supersymmetry transformations (setting $\zeta^m = 0$, $\omega = 0$)

$$\xi^m = i(\theta\gamma^m\bar\zeta) - \theta\bar\theta(\zeta\bar\lambda^m), \tag{100}$$

$$\xi^\mu = \zeta^\mu + i(\theta\gamma^b\bar\zeta)\chi_b{}^\mu + \frac{1}{2}\theta\bar\theta\left[\lambda^2\zeta^\mu + \frac{i}{2}c_b(\zeta\gamma^b)^\mu\right], \tag{101}$$

$$W = -A(\theta\gamma^5\bar\zeta) - i\rho_b(\theta\gamma^b\bar\zeta) - \theta\bar\theta\left[\epsilon^{mn}(\zeta\widetilde{\nabla}_m\bar\chi_n) + c_b(\zeta\gamma^5\bar\lambda^b) - i\lambda^2(\zeta\gamma^5\bar\chi)\right]. \tag{102}$$

We see that the bosonic vector field $\zeta^m(x)$ only enters as the zeroth component of $\xi^m(z)$. Notice that with respect to the Lorentz boost the coordinate θ^μ changes as if it were a spinor in the anholonomic basis. This is a consequence of the 'gauge' condition (42) which simply implies that in the zeroth order of θ the spinor components in the holonomic and in the anholonomic basis are identified and, therefore, must be transformed simultaneously. The local supersymmetry transformation parameters $\zeta^\mu(x)$ appear in a complicated manner and produce nontrivial transformations in the bosonic subspace as well as a local Lorentz boost. Notice also that the parameters only depend on the supergravity multiplet $\mathcal{E}$. This means that consistent supergravity can be and was constructed in terms of that quantity. The transformation rules for the supergravity multiplet are obtained by considering the variations $\delta E^{(0)}{}_a{}^m$, $\delta E^{(0)}{}_a{}^\mu$, $\delta\Omega^{(1)}{}_\alpha$ under the

transformations (97)–(102). For bosonic diffeomorphisms and Lorentz boosts one verifies the desired transformation laws for fields according to their representations

$$\delta e_a{}^m = \zeta^n \partial_n e_a{}^m - e_a{}^n \partial_n \zeta^m - \omega \epsilon_a{}^b e_b{}^m, \tag{103}$$

$$\delta \chi_a{}^\mu = \zeta^n \partial_n \chi_a{}^\mu - \omega \epsilon_a{}^b \chi_b{}^\mu - \frac{1}{2}\omega (\chi_a \gamma^5)^\mu, \tag{104}$$

$$\delta A = \zeta^m \partial_m A. \tag{105}$$

This a posteriori justifies the gauge fixing procedure in section 3, restricting general superdiffeomorphisms. Under local supersymmetry the supergravity supermultiplet $\mathcal{E}$ transforms as

$$\delta e_a{}^m = 2i(\zeta \gamma^m \bar{\chi}_a), \tag{106}$$

$$\delta \chi_a{}^\mu = -\widetilde{\nabla}_a \zeta^\mu - 2i(\chi \bar{\lambda}_a)\zeta^\mu - 2i(\lambda_b \bar{\chi}_a)(\zeta \gamma^b)^\mu - \frac{i}{2} A(\zeta \gamma_a)^\mu \tag{107}$$

$$\delta A = 2\epsilon^{mn}(\zeta \gamma^5 \widetilde{\nabla}_m \bar{\chi}_n) + 2i\lambda^2(\zeta \bar{\chi}) - 2iA(\zeta \bar{\chi}). \tag{108}$$

Transformation rules for the connection supermultiplet $\Omega_a = \{\omega_a, u_a{}^\mu, \upsilon_a\}$ may be read off from the variation of Ω_a. These rules appear to be rather complicated. One of the reasons for this is that this supermultiplet consists of connections with more complicated transformational properties than tensors. It turns out that the introduction of a new torsion supermultiplet $T_a = \{t_a, \tau_a{}^\mu, s_a\}$ is more convenient than to work in terms of Ω_a. The field t_a is given by (89), whereas $\tau_a{}^\mu$ and s_a are the first and second order components of the torsion superfield T_a as defined by (90) and (91). The new fields are related to the old ones by algebraic invertible equations and thus both supermultiplets are equivalent. It should be admitted that in terms of T_a the interaction with matter superfields is very likely to be more complicated in higher orders of θ (because in the zeroth order we still keep the Lorentz connection as an independent variable) but this is beyond the scope of the present paper. In any case the transformation rules of Ω can be deduced by the interested reader from the formulas of the present paper.

To obtain the transformation rules for T_a we consider the variation

$$\delta T_a = \xi^m \partial_m T_a + \xi^\mu \partial_\mu T_a - W \epsilon_a{}^b T_b$$

and arrive at the familiar transformations of bosonic diffeomorphisms and Lorentz boosts

$$\delta t_a = \zeta^n \partial_n t_a - \omega \epsilon_a{}^b t_b, \tag{109}$$

$$\delta \tau_a{}^\mu = \zeta^n \partial_n \tau_a{}^\mu - \omega \epsilon_a{}^b \tau_b{}^\mu - \frac{1}{2}\omega(\tau_a \gamma^5)^\mu, \tag{110}$$

$$\delta s_a = \zeta^n \partial_n s_a - \omega \epsilon_a{}^b s_b. \tag{111}$$

Under local supersymmetry T_a transforms as

$$\delta t_a = (\zeta\bar{\tau}_a) \tag{112}$$

$$\begin{aligned}\delta \tau_a{}^\mu = &-i(\zeta\gamma^b)^\mu \left[\widetilde{\nabla}_b t_a - 4i\epsilon_a{}^c t_c(\lambda_b\gamma^5\bar{\chi}) + (\chi_b\bar{\tau}_a)\right] \\ &-(\zeta\gamma^5)^\mu A\epsilon_a{}^b t_b + \zeta^\mu s_a,\end{aligned} \tag{113}$$

$$\begin{aligned}\delta s_a = &-2(\zeta\bar{\lambda}^b)\widetilde{\nabla}_b t_a + 2\epsilon^{mn}(\zeta\widetilde{\nabla}_m\bar{\chi}_n)\epsilon_a{}^b t_b - 2i(\zeta\gamma^5\bar{\chi})\lambda^2\epsilon_a{}^b t_b + i(\zeta\gamma^b\widetilde{\nabla}_b\bar{\tau}_a) \\ &+(\zeta\bar{\tau}_a)\lambda^2 + 4\epsilon_a{}^b(\zeta\gamma^c\bar{\tau}_b)(\lambda_c\gamma^5\bar{\chi}) + A\epsilon_a{}^b(\zeta\gamma^5\bar{\tau}_b) - 2i(\zeta\bar{\chi})s_a.\end{aligned} \tag{114}$$

Because of its importance we also add the transformation of the bosonic Lorentz connection, although it is implied by (112),

$$\begin{aligned}\delta\omega_a = &\epsilon_a{}^b(\zeta\bar{\tau}_b) + 2i\epsilon^{mn}(\zeta\gamma_a\widetilde{\nabla}_m\bar{\chi}_n) + 2ic_b(\zeta\gamma^b\gamma^5\bar{\chi}_a) \\ &+8(\zeta\gamma^5\bar{\lambda}_a)\chi^2 + 2(\zeta\gamma_a\gamma^5\bar{\chi})\lambda^2 - 2A(\zeta\gamma^5\bar{\lambda}_a).\end{aligned} \tag{115}$$

We see that the parameter $\zeta^m(x)$ exactly produces the general coordinate transformations (diffeomorphisms) of the bosonic subspace. The Lorentz boost with parameter $\omega(x)$ not only rotates the anholonomic indices of vectors and spinors but also the holonomic spinor indices. Thus we have identified the Lorentz boost in the tangent space with the Lorentz subgroup entering the group of general coordinate transformations in superspace. It seems remarkable that also for our generalized supergravity the transformation of the supergravity supermultiplet does not involve extra fields and remains the same as for vanishing bosonic torsion [6]. The Lorentz superconnection Ω_a as a superfield or the trace of the supertorsion T_a form separate supermultiplets whose transformation rules contain the supergravity multiplet. The same is known to happen if one adds additional matter superfields [25]. Then the transformation of its components will involve the supergravity supermultiplet because it explicitly enters the transformation parameters (100)–(102).

7 Supergravity Lagrangians

The construction of generic supergravity Lagrangians within the superspace approach is simple but implies lengthy calculations when one desires to write them in terms of all the fields contained in the supermultiplets. A functional

$$I = \int d^2x d^2\theta \; E \; L(x,\theta), \tag{116}$$

($E = \mathrm{sdet}\, E_M{}^A$ denotes the Berezinian (superdeterminant) of the supervierbein [26, 27] and L is an arbitrary scalar superfield built from supermultiplets) after integration over θ yields a supergravity model written for a set of fields over two-dimensional space-time. If a supermanifold has a boundary or a nontrivial

topology then the usual integration rule over θ must be modified [22]. In our case we have two supermultiplets united in two superfields S and T_a. The anholonomic indices are transformed only under a Lorentz boost, and Latin and Greek indices do not mix. Thus one may construct scalar Lagrangians by contracting Latin and Greek indices separately. There are, of course, an infinite number of choices. For example, using the definition of the scalar curvature of superspace (35) and the explicit form of its components (83)–(85) one easily finds for the supercurvature invariant (35)

$$R = \epsilon^{\alpha\beta}\nabla_\beta\nabla_\alpha S - 2S^2 + 2\nabla_a T^a - 2T_a T^a - 2S. \tag{117}$$

It is important to note that any term on the right hand side is a scalar superfield and can be chosen as a Lagrangian.

Let us consider some of the simplest Lagrangians, to be obtained by special choices of L in (116). Using the definition

$$E = \operatorname{sdet} E_M{}^A = \frac{\det(E_m{}^a - E_m{}^\beta E^{-1}{}_\beta{}^\nu E_\nu{}^a)}{\det E_\mu{}^\alpha} \tag{118}$$

and the expression for the supervierbein in terms of the supergravity supermultiplet (73)–(76) we have

$$E = \det e_m{}^a \left[1 + 2i(\theta\bar{\chi}) + \frac{1}{2}\theta\bar{\theta}(2\chi^2 + \lambda^2 + A)\right]. \tag{119}$$

Integrating θ in (116) produces the second order component of the product of the superdeterminant and the scalar superfield chosen as a Lagrangian. The simplest example $L = 1$ would correspond to the cosmological constant in ordinary gravity. In supergravity, with the superdeterminant alone, we obtain

$$\mathcal{L}_1 = A + 2\chi^2 + \lambda^2. \tag{120}$$

Here and below in such Lagrangians in 2d space-time we drop the factor $e = \det e_m{}^a$ for brevity, thus $I_1 = \int d^2x\, e\, \mathcal{L}_1$. This Lagrangian by itself has only trivial solutions but may yield nontrivial contributions if added to other Lagrangians. We see that no cosmological constant can be added to a supergravity model in this way.

For $L = S$ one arrives at

$$\mathcal{L}_S = -\frac{1}{2}\widetilde{R} - 4i\widetilde{\nabla}_a(\lambda^a\bar{\chi}). \tag{121}$$

where $\widetilde{R}$ and $\widetilde{\nabla}$ are defined in appendix B. Thus this Lagrangian multiplied by e equals to a total derivative. Therefore, the 'minimal' supergravity in two dimensions is as trivial as the bosonic Hilbert-Einstein action, represented by the

first term in (121). However, the components of the supergravity multiplet are essential for constructing interactions with matter superfields. For example, they are of particular relevance to superstring theory.

Other scalar superfields provide nontrivial models with second order equations of motion. For $L = S^2$ we have

$$\begin{aligned}\mathcal{L}_{S^2} = &-A\widetilde{R} - A^3 + 4\epsilon^{mn}\epsilon^{pr}((\widetilde{\nabla}_m\chi_n)(\widetilde{\nabla}_p\bar{\chi}_r)) + 8i(A-\lambda^2)\epsilon^{mn}(\chi\gamma^5\widetilde{\nabla}_m\bar{\chi}_n) \\ &- 8iA\widetilde{\nabla}_a(\lambda^a\bar{\chi}) - 8A\chi^2\lambda^2 - A^2(\chi^a\bar{\chi}_a).\end{aligned} \tag{122}$$

The term $\nabla^\alpha\nabla_\alpha S$ is also contained in the expression (117) for R. Taken as a Lagrangian L the related $\mathcal{L}_{\nabla^2 S}$ turns out to be proportional to (122) up to a total divergence.

Eq. (122) represents a supergravity theory with vanishing bosonic torsion. Therefore, we could have obtained it without the extension discussed in our present paper. The Rarita-Schwinger field appears with first and second derivatives. On the other hand, the scalar field A is nondynamical. Due to the linear term $A\widetilde{R}$, however, by the conformal transformation $e_m{}^a \to Ae_m{}^a$ of the zweibein a kinetic term for A may be produced, and $-(\ln|A|)/2 = \phi$ may be interpreted as a dilaton field [13]. In that case from the bosonic part of (122) alone theories with interesting highly nontrivial singularity properties may be obtained. A different approach to supersymmetric dilaton gravity is adopted in [28, 25], where an extra dilaton superfield is introduced. In our case the bosonic dilaton field arises from the scalar component of the gravity supermultiplet.

In a similar way one obtains the two simplest Lagrangians containing the torsion supermultiplet T_a with at most second order e. o. m.-s. $L = T^2$ in (116) leads to (cf. (89))

$$\begin{aligned}\mathcal{L}_{T^2} = &(A + 2\chi^2 + \lambda^2)\hat{t}_a\hat{t}^a + 2s^a t_a - (\tau^a\bar{\tau}_a) \\ &- 4i(\chi\bar{\tau}^a)\, t_a - 8iA\hat{t}^a(\lambda_a\bar{\chi}) + 8A\chi^2\lambda^2.\end{aligned} \tag{123}$$

The Lagrangian for $\nabla_a T^a$ turns out to be the same up to a total divergence. Therefore, their difference does not contribute to the analog of the Hilbert-Einstein action L_R in superspace, constructed with (117) as a Lagrangian. Thus this entire Lagrangian in superspace is proportional to (122) only. The Lagrangian (123) leads to the constraint $T_a = 0$, i. e. the Lagrangians (120)–(122) exhaust the set of nontrivial ones with not more than two derivatives in the e. o. m.-s for the fields $e_a{}^m$, ω_a and their supergravity partners. We are thus led to the conclusion that in any construction of L using the superfields S and T_a and requiring at most first derivatives of the supergravity Cartan variables in $\mathcal{L}$, bosonic torsion has to vanish after all.

The construction of the supergravity models above is based on the action principle in superspace. Another possibility to get a supersymmetric extension of a given bosonic model consists in the generalization of the equations of motion to

superspace. For example, the super Liouville model can be naturally formulated in terms of the supergravity multiplet. It is well known that two-dimensional constant curvature gravity [29], $\widetilde{R} = \text{const}$, in the conformal gauge reduces to the Liouville equation. The super extension of this model is given by the invariant equation

$$S = C = \text{const},$$

or in components

$$\begin{aligned} A &= C, \\ \epsilon^{mn}(\gamma^5 \widetilde{\nabla}_m \bar{\chi}_n)_\alpha + i\lambda^2 \bar{\chi}_\alpha - iC\bar{\chi}_\alpha &= 0, \\ \frac{1}{2}\widetilde{R} - 4i\epsilon^{mn}(\chi\gamma^5\widetilde{\nabla}_m\bar{\chi}_n) + 4i\widetilde{\nabla}_a(\lambda^a\bar{\chi}) + 4\chi^2\lambda^2 + C(\chi^a\bar{\chi}_a) + C^2 &= 0. \end{aligned}$$

The field A is constant due to the first equation, and the last two of this super-covariant system of equations may be rewritten in equivalent form

$$-(\gamma^a\widetilde{\nabla}_a\bar{\chi})_\alpha - \widetilde{\nabla}_a\bar{\lambda}^a{}_\alpha + i\lambda^2\bar{\chi}_\alpha - iC\bar{\chi}_\alpha = 0, \tag{124}$$

$$\widetilde{R} + 8i\widetilde{\nabla}_a(\lambda^a\bar{\chi}) + 2C(2\chi^2 + \lambda^2) + 2C^2 = 0. \tag{125}$$

These equations reduce to the constant curvature gravity in the absence of the Rarita-Schwinger field. It seems to be the simplest nontrivial supergravity model in two dimensions. This super extension of the Liouville model differs from the known generalizations. The effective superspace action for a superstring off the critical dimension may be also considered as the super extension of the Liouville model [30]. In that case the action depends on the extra scalar superfield. Another super extension of the Liouville model may be found in [31] where the Liouville action is generalized without incorporating the general coordinate invariance.

In order to explore possibilities of some higher order Lagrangians it is sufficient for a first orientation to determine their bosonic parts. Of course, the supplementing superfields can be introduced in all cases in a straightforward manner.

For vanishing fermionic fields the inverse supervielbein (67)–(70) and the

Lorentz superconnection (71), (72) take a particularly simple form

$$E_a{}^m = e_a{}^m, \tag{126}$$

$$E_a{}^\mu = -\frac{i}{2}A(\theta\gamma_a)^\mu + \frac{1}{2}\widetilde{\omega}_a(\theta\gamma^5)^\mu \tag{127}$$

$$E_\alpha{}^m = i(\gamma^m\bar{\theta})_\alpha, \tag{128}$$

$$E_\alpha{}^\mu = \delta_\alpha{}^\mu - \frac{1}{2}\theta\bar{\theta}\left[\frac{1}{2}A\delta_\alpha{}^\mu + \frac{i}{2}c_b\gamma^b{}_\alpha{}^\mu\right], \tag{129}$$

$$\Omega_a = \omega_a + \frac{1}{2}\theta\bar{\theta}\epsilon_a{}^b(s_b + \partial_b A), \tag{130}$$

$$\Omega_\alpha = A(\theta\gamma^5\epsilon)_\alpha - i\widetilde{\omega}_b(\theta\gamma^b\epsilon)_\alpha. \tag{131}$$

and the corresponding supervielbein (73)–(76) becomes

$$E_m{}^a = e_m{}^a + \frac{1}{2}\theta\bar{\theta}Ae_m{}^a, \tag{132}$$

$$E_m{}^\alpha = \frac{i}{2}A(\theta\gamma_m)^\alpha - \frac{1}{2}\widetilde{\omega}_m(\theta\gamma^5)^\alpha, \tag{133}$$

$$E_\mu{}^a = i(\theta\gamma^a\epsilon)_\mu, \tag{134}$$

$$E_\mu{}^\alpha = \delta_\mu{}^\alpha\left(1 - \frac{1}{4}\theta\bar{\theta}A\right). \tag{135}$$

Also the superfields S and T_a simplify greatly:

$$S = A - \frac{1}{2}\theta\bar{\theta}\left[\frac{1}{2}\widetilde{R} + A^2\right], \tag{136}$$

$$T_a = t_a + \frac{1}{2}\theta\bar{\theta}s_a. \tag{137}$$

Together with the superdeterminant

$$E = e\left(1 + \frac{1}{2}\theta\bar{\theta}A\right) \tag{138}$$

the bosonic parts of the possible supergravity Lagrangians (116), constructed from scalar and vector superfields (136) and (137) are determined easily and can be generalized to obtain other models which permit a direct 'dilatonization'. Consider an arbitrary power k of the scalar superfield S^k. The corresponding bosonic Lagrangian is

$$\mathcal{L}_{S^k} = -\frac{k}{2}A^{k-1}\widetilde{R} - (k-1)A^{k+1}.$$

For negative scalar curvature the field A can be eliminated using its equation of motion, and the Lagrangian becomes

$$\mathcal{L}_{S^k} \approx (-\widetilde{R})^{\frac{k+1}{2}}.$$

This Lagrangian yields a large nontrivial class of gravity models in two dimensions. All of them are integrable [13] (that is one can write down a general solution to the equations of motion and even analyse the corresponding unique topology of the space-time) and can be made locally supersymmetric. Identifying now $kA^{k-1} = -\exp(-2\phi)/2$, again A (or ϕ) can be made dynamical by a conformal transformation of ${e_m}^a$ as in the case $k = 1$. Of course, also polynomials and even arbitrary functions of S could be considered in L [8]. Also conformal transformations by a superfield may provide a supersymmetric 'dilatonization' [8]. The bosonic part of those theories then becomes just a generic one of covariant PSM-models [11] with vanishing torsion. Supplementing the supersymmetric part immediately provides again supergravity extensions.

Let us briefly discuss other possible generalizations. In the examples above the scalar field A acted essentially as an auxiliary field because it entered the Lagrangian without kinetic term and could be eliminated by solving its algebraic equation of motion. However, in general, also a supersymmetric kinetic term for A may exist. Consider, for example, the scalar superfield $\nabla^\alpha S \nabla_\alpha S$. The corresponding Lagrangian has the form

$$\mathcal{L}_{\nabla^\alpha S \nabla_\alpha S} = 2\eta^{ab}\partial_a A \partial_b A + \frac{1}{2}(\widetilde{R} + 2A^2)^2.$$

The appearance of the (torsionless) scalar curvature in the second power yields higher derivatives. The other possible candidate with second derivatives for A is

$$\mathcal{L}_{\nabla^a S \nabla_a S} = -3A\partial^a A \partial_a A - \partial^a A \partial_a \widetilde{R}.$$

Another example, involving nonvanishing bosonic torsion, is the (for the zweibein only) higher derivative Lagrangian containing a kinetic term for the Lorentz connection ω_a

$$\mathcal{L}_{\nabla^\alpha T^a \nabla_\alpha T_a} = 2\widetilde{\nabla}^a \hat{t}^b \widetilde{\nabla}_a \hat{t}_b + 2s^a s_a + 2A^2 \hat{t}^a \hat{t}_a.$$

Such a kinetic term for the Lorentz connection ω_a is also contained in a Lagrangian from $L = R^2$. For simplicity we abbreviate the scalar supercurvature by extracting the last term in (117)

$$\underline{R} = -\epsilon^{ab} F_{ba} = R + 2S.$$

A simple calculation gives the bosonic Lagrangian (quantities with hats are built with the x-space connection ω_m, cf. appendix B)

$$\mathcal{L}_{\underline{R}^2} = 3A\widehat{R}^2 + 4\widehat{R}\widehat{\nabla}^a\widehat{\nabla}_a A + 4A^3\widehat{R} + 4\widetilde{\nabla}_a s^a + 8\hat{t}^a \partial_a A.$$

There is, of course, an infinite number of other higher order bosonic Lagrangians which can be made locally supersymmetric, also including higher powers of torsion.

8 Summary and Outlook

The generalization of rigid supersymmetry to generalized theories of supergravity does not necessitate the validity of the bosonic torsion constraint. Dropping the latter in 1+1 space-time we solve the minimal set of constraints (39) for $N = (1,1)$ superspace. The computational problems which would occur following the approach of the seminal work by Howe [6] are greatly reduced by working in terms of the *inverse* supervierbein and the Lorentz superconnection. After conventional gauge fixing of superdiffeomorphisms and Lorentz boosts it is possible to *first* solve the constraints. The inverse supervierbein turns out to be expressed only in terms of a supergravity multiplet $\mathcal{E} = (e_a{}^m, \chi_a{}^\mu, A)$ consisting of the zweibein, a Rarita-Schwinger field and a spinor field contained in the (reducible) field $\chi_a{}^\mu$ and a scalar A. The Lorentz superconnection in turn, beside the components of $\mathcal{E}$ has arbitrary components expressible either as a supermultiplet of connections $\Omega_a = (\omega_a, u_a{}^\mu, v_a)$ or, alternatively, by a torsion multiplet T_a of the same structure. As a consequence of our approach, the Bianchi identities are fulfilled identically. Nevertheless, they are very useful in order to find out how the components of $\mathcal{E}$ may be summarized in a scalar superfield S; T_a may be interpreted as the trace of the supertorsion. Since the component fields are found to transform with respect to the correct local diffeomorphisms, local Lorentz and supersymmetry transformations, a generic superfield Lagrangian is a superscalar built from S and T_a, multiplied by the superdeterminant of $E_M{}^A$. Considering the simplest example which leads to at most second order equations of motion for the fields we find that nontrivial Lagrangians then are restricted to the case $T_a = 0$. Therefore no action of this type may produce the immediate generalization of two-dimensional gravity with torsion [10]. However, 2d theories with arbitrary powers in bosonic curvature [8] — and vanishing torsion — in our approach are readily extended to completely supersymmetric ones with all superfield-partners included. Such theories also allow 'dilatonization', i. e. by conformal transformation of the zweibein or metric involving the — nondynamical — scalar field A in S, globally quite different dilaton theories with A related to the dilaton field may be produced [13]. There are even many types of higher derivative theories with nonvanishing torsion ($T_a \neq 0$) to be exploited in further work. From recent results on general classical and quantum solutions for any such theory in the purely bosonic case [11, 13, 19] it may be conjectured that their integrability may be extended to these supergravity generalizations, allowing for exactly solvable models for black holes including matter in the form of superpartners of the zweibein and the Lorentz connection. Also the relation to the first order formulation in [11, 13, 19] needs clarification. An interesting generalisation to the heterotic supergeometry [32] may be relevant to heterotic string theory. These are some of the topics we intend to tackle in further work.

A Conventions of Spinor-Space and Spinors

Of course, the properties of Clifford algebras and spinors in any number of dimensions (including $d = 1+1$) are well-known, but in view of the tedious calculations required in our present work we include this appendix in order to prevent any misunderstandings of our results and facilitate the task of the intrepid reader who wants to redo derivations.

The γ matrices which are the elements of the Clifford algebra defined by the relation

$$\gamma^a \gamma^b + \gamma^b \gamma^a = 2\eta^{ab}, \quad \eta_{ab} = \eta^{ab} = \mathrm{diag}(+-), \tag{139}$$

are represented by two-dimensional matrices

$$\gamma^0{}_\alpha{}^\beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1{}_\alpha{}^\beta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{140}$$

As indicated, the lower index is assumed to be the first one. The spinor indices are often suppressed assuming the summation from 'ten to four'. The generator of a Lorentz boost (hyperbolic rotation) has the form

$$\sigma^{ab} = \frac{1}{2}(\gamma^a \gamma^b - \gamma^b \gamma^a) = \epsilon^{ab}\gamma^5, \tag{141}$$

where

$$\gamma^5 = -\gamma^0\gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (\gamma^5)^2 = 1, \tag{142}$$

and

$$\epsilon_{ab} = -\epsilon^{ab} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{143}$$

is the totally antisymmetric tensor with vector indices obeying

$$\epsilon^{ab}\epsilon_{cd} = -\delta_c{}^a \delta_d{}^b + \delta_c{}^b \delta_d{}^a, \quad \epsilon^{ab}\epsilon_{bd} = \delta_d{}^a, \quad \epsilon^{ab}\epsilon_{ba} = 2, \tag{144}$$

where $\delta_a{}^b = \delta_a^b$ denotes the Kronecker symbol. In two dimensions the γ-matrices satisfy the relation

$$\gamma^a \gamma^b = \eta^{ab} + \epsilon^{ab}\gamma^5, \tag{145}$$

which is equivalent to the definition (139). The following formulas are frequently used in our calculations:

$$\begin{aligned} \gamma^a\gamma_a = 2, &\qquad \gamma^a\gamma^b\gamma_a = 0, \\ \gamma^a\gamma^5 + \gamma^5\gamma^a = 0, &\qquad \gamma^a\gamma^5 = \gamma^b\epsilon_b{}^a, \\ \mathrm{tr}(\gamma^a\gamma^b) = 2\eta^{ab}. & \end{aligned} \tag{146}$$

As usual the trace of the product of an odd number of γ-matrices vanishes.

In two dimensions the γ-matrices satisfy the Fierz identity

$$2\gamma^a{}_\alpha{}^\gamma\gamma^b{}_\beta{}^\delta = \gamma^a{}_\alpha{}^\delta\gamma^b{}_\beta{}^\gamma + \gamma^b{}_\alpha{}^\delta\gamma^a{}_\beta{}^\gamma + \\ + \eta^{ab}(\delta_\alpha{}^\delta\delta_\beta{}^\gamma - \gamma^5{}_\alpha{}^\delta\gamma^5{}_\beta{}^\gamma - \gamma^c{}_\alpha{}^\delta\gamma_{c\beta}{}^\gamma) + \epsilon^{ab}(\gamma^5{}_\alpha{}^\delta\delta_\beta{}^\gamma - \delta_\alpha{}^\delta\gamma^5{}_\beta{}^\gamma), \quad (147)$$

which can be checked by direct calculation. Different contractions of it with γ-matrices then yield different but equivalent versions

$$2\delta_\alpha{}^\gamma\delta_\beta{}^\delta = \delta_\alpha{}^\delta\delta_\beta{}^\gamma + \gamma^5{}_\alpha{}^\delta\gamma^5{}_\beta{}^\gamma + \gamma^a{}_\alpha{}^\delta\gamma_{a\beta}{}^\gamma, \quad (148)$$

$$2\gamma^5{}_\alpha{}^\gamma\gamma^5{}_\beta{}^\delta = \delta_\alpha{}^\delta\delta_\beta{}^\gamma + \gamma^5{}_\alpha{}^\delta\gamma^5{}_\beta{}^\gamma - \gamma^a{}_\alpha{}^\delta\gamma_{a\beta}{}^\gamma, \quad (149)$$

$$\gamma^a{}_\alpha{}^\gamma\gamma_{a\beta}{}^\delta = \delta_\alpha{}^\delta\delta_\beta{}^\gamma - \gamma^5{}_\alpha{}^\delta\gamma^5{}_\beta{}^\gamma, \quad (150)$$

which allow to manipulate third and higher order monomials in spinors. Notice that equation (145) is also the consequence of (147).

The totally antisymmetric tensor and the Minkowskian metric satisfy the Fierz-type identity in two dimensions

$$\eta_{ab}\epsilon_{cd} + \eta_{da}\epsilon_{bc} + \eta_{cd}\epsilon_{ab} + \eta_{bc}\epsilon_{da} = 0, \quad (151)$$

which allows to make rearrangements in third and higher order monomials of Lorentz vectors.

A Dirac spinor in two dimensions, forming an irreducible representation for the full Lorentz group including space and time reflections, has two complex components. We write it — in contrast to the usual convention in field theory, but in agreement with conventional superspace notations — as a row

$$\psi^\alpha = (\psi^1, \psi^2). \quad (152)$$

In our notation the first and second components of a Dirac spinor correspond to right and left chiral Weyl spinors $\psi^{(\pm)}$, respectively,

$$\psi^{(\pm)} = \psi\frac{1 \pm \gamma^5}{2}, \quad \psi^{(\pm)}\gamma^5 = \pm\psi^{(\pm)}, \quad (153)$$

where

$$\psi^{(+)} = (\psi^1, 0), \qquad \psi^{(-)} = (0, \psi^2). \quad (154)$$

Here matrices act on spinors form the right according to the usual multiplication law. All spinors are always assumed to be anticommuting variables. The notation with upper indices is a consequence of our convention to contract indices, together with the usual multiplication rule for matrices. Under the Lorentz boost by the parameter ω spinors transform as

$$\psi'^\alpha = \psi^\beta S_\beta{}^\alpha, \quad (155)$$

where

$$S_\beta{}^\alpha = \delta_\beta{}^\alpha \cosh\frac{\omega}{2} - \gamma^5{}_\beta{}^\alpha \sinh\frac{\omega}{2} = \begin{pmatrix} e^{-\omega/2} & 0 \\ 0 & e^{+\omega/2} \end{pmatrix}, \tag{156}$$

when the Lorentz boost of a vector is given by the matrix

$$S_b{}^a = \delta_b{}^a \cosh\omega + \epsilon_b{}^a \sinh\omega = \begin{pmatrix} \cosh\omega & -\sinh\omega \\ -\sinh\omega & \cosh\omega \end{pmatrix}. \tag{157}$$

By (156), (157) the γ-matrices are invariant under simultaneous transformation of Latin and Greek indices. This requirement fixes the relative factors in the bosonic and fermionic sectors of the Lorentz generator (24).

Dirac conjugation is defined in the usual way and is written as a column

$$\bar\psi_\alpha = \psi^{\beta *} g_{\beta\alpha} = \begin{pmatrix} \psi^{2*} \\ \psi^{1*} \end{pmatrix}, \quad g_{\beta\alpha} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{158}$$

where star $*$ denotes complex conjugation. Here we see that the role of the matrix γ^0 is twofold. First, it is an operator in the spinor space and thus has one lower and one upper index (140). The same matrix defines the metric in spinor space. Therefore it is written in (158) with two lower indices and for clarity is denoted by the different symbol $g_{\alpha\beta}$.

In our conventions bilinear forms of spinors ψ and χ appear as

$$\psi\Gamma\bar\chi = \psi^\alpha \Gamma_\alpha{}^\beta \bar\chi_\beta, \tag{159}$$

where Γ denotes any polynomial of the unit and the γ-matrices. Under a Lorentz boost the bilinear forms $\psi\gamma^a\bar\chi$ and $\psi\bar\chi$, $\psi\gamma^5\bar\chi$ transform as a vector and as scalars.

Among the discrete transformations the parity transformation (supposed to be linear)

$$P\colon \psi^\alpha \to \psi_p^\alpha = \psi^\beta P_\beta{}^\alpha \tag{160}$$

is uniquely defined by

$$\begin{aligned} \psi_p\bar\chi_p &= \psi\bar\chi, \\ \psi_p\gamma^0\bar\chi_p &= \psi\gamma^0\bar\chi, \\ \psi_p\gamma^1\bar\chi_p &= -\psi\gamma^1\bar\chi, \end{aligned} \tag{161}$$

up to an arbitrary complex number with unit modulus. We choose it to be real

$$P = \gamma^0. \tag{162}$$

It can be checked easily that

$$\psi_p\gamma^5\bar\chi_p = -\psi\gamma^5\bar\chi, \tag{163}$$

i. e. is a pseudoscalar.

By definition charge (or Majorana) conjugation relates a spinor to its Dirac conjugate

$$C\colon \psi^\alpha \to \psi_c^\alpha = C^{\alpha\beta}\bar\psi_\beta, \tag{164}$$

$$\bar\psi_{c\alpha} = C^{-1}_{\alpha\beta}\psi^\beta. \tag{165}$$

The last equation is needed in order that the square of charge conjugation results in the identical transformation. Because Dirac conjugation is already defined by (158) the charge conjugation matrix must satisfy the relation

$$C^{-1}_{\alpha\beta} = C^{\dagger\gamma\delta} g_{\delta\alpha} g_{\gamma\beta}, \tag{166}$$

where the cross denotes hermitian conjugation. Requiring

$$\psi_c\bar\psi_c = \psi\bar\psi, \tag{167}$$

$$\psi_c\gamma^a\bar\psi_c = -\psi\gamma^a\bar\psi, \tag{168}$$

in order to preserve the sign of mass and to invert the sign of the electric charge, already equation (167) defines the charge conjugation matrix up to an arbitrary complex number of unit modulus which we fix to unity

$$C^{\alpha\beta} = \epsilon^{\alpha\beta}, \quad C^{-1}_{\alpha\beta} = -\epsilon_{\alpha\beta}, \tag{169}$$

where

$$\epsilon_{\alpha\beta} = \epsilon^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{170}$$

is the totally antisymmetric tensor with spinor indices. It has the properties

$$\epsilon^{\alpha\beta}\epsilon_{\gamma\delta} = \delta_\gamma{}^\alpha\delta_\delta{}^\beta - \delta_\gamma{}^\beta\delta_\delta{}^\alpha, \quad \epsilon^{\alpha\beta}\epsilon_{\beta\gamma} = -\delta_\gamma{}^\alpha, \quad \epsilon^{\alpha\beta}\epsilon_{\beta\alpha} = -2. \tag{171}$$

With (169) relations (166), (168) are verified. Under charge conjugation a pseudoscalar changes its sign

$$\psi_c\gamma^5\bar\psi_c = -\psi\gamma^5\bar\psi. \tag{172}$$

In the Majorana spinor — defined by the requirement that its Dirac conjugate equals the charge conjugate —

$$\bar\psi_\alpha = C^{-1}_{\alpha\beta}\psi^\beta \quad \Leftrightarrow \quad \psi^{*\alpha} = \psi^\beta\gamma^5{}_\beta{}^\alpha \tag{173}$$

the first component is real while the second is purely imaginary

$$\psi^{1*} = \psi^1, \quad \psi^{2*} = -\psi^2. \tag{174}$$

For Majorana spinors the bilinear form

$$\psi\bar{\chi} = \psi^\alpha \chi^\beta \epsilon_{\beta\alpha}, \tag{175}$$

can be considered to be defined by the metric $\epsilon_{\alpha\beta}$ in spinor space. It has no definite parity because the definition of a Majorana spinor (173) implies the γ^5 matrix. The bilinear combinations of Majorana spinors have the properties

$$\begin{array}{rclcll}
\psi\bar{\chi} & = & \chi\bar{\psi} & = & -\psi^1\chi^2 + \psi^2\chi^1 & \text{real} \\
\psi\gamma^5\bar{\chi} & = & -\chi\gamma^5\bar{\psi} & = & -\psi^1\chi^2 - \psi^2\chi^1 & \text{real} \\
\psi\gamma^0\bar{\chi} & = & -\chi\gamma^0\bar{\psi} & = & \psi^1\chi^1 - \psi^2\chi^2 & \text{imaginary} \\
\psi\gamma^1\bar{\chi} & = & -\chi\gamma^1\bar{\psi} & = & \psi^1\chi^1 + \psi^2\chi^2 & \text{imaginary,}
\end{array}$$

where one should remember that complex conjugation changes the order of anticommuting variables. In particular, for Majorana spinors

$$\psi\gamma^a\bar{\psi} = 0, \quad \psi\gamma^5\bar{\psi} = 0 \tag{176}$$

the only nonvanishing quadratic form being $\psi\bar{\psi} = \psi^\alpha\psi^\beta\epsilon_{\beta\alpha}$. As a consequence of equation (145) a quadratic form in Majorana spinors with an arbitrary odd number of γ-matrices is always zero.

The field $\chi^{a\alpha}$ has one vector and one spinor index. We assume that for each a it is a Majorana spinor. Therefore it has two real and two purely imaginary components forming a reducible representation of the Lorentz group. In many applications it becomes extremely useful to work with its Lorentz covariant decomposition

$$\chi^a = \chi\gamma^a + \lambda^a, \tag{177}$$

where

$$\chi = \frac{1}{2}\chi^a\gamma_a, \quad \lambda^a = \frac{1}{2}\chi^b\gamma^a\gamma_b. \tag{178}$$

The spinor χ^α and the spin-vector $\lambda_a{}^\alpha$ form irreducible representations of the Lorentz group and each of them has two independent components. The spin-vector λ_a satisfies the Rarita-Schwinger condition

$$\lambda_a\gamma^a = 0 \tag{179}$$

valid for such a field. In two dimensions equation (179) may be written in equivalent forms

$$\lambda_a\gamma_b = \lambda_b\gamma_a \quad \text{or} \quad \epsilon^{ab}\lambda_a\gamma_b = 0. \tag{180}$$

If one chooses the $\lambda^{0\alpha}$ components as independent ones then the components of $\lambda^{1\alpha}$ can be found from (179) to be

$$\lambda^{0\alpha} = (\lambda^{01}, \lambda^{02}), \quad \lambda^{1\alpha} = (\lambda^{01}, -\lambda^{02}).$$

It is important to note that as a consequence any cubic or higher monomial of the (anticommuting) χ or λ_a vanishes identically. The field λ_a satisfies further the useful relation

$$\epsilon_a{}^b \lambda_b = \lambda_a \gamma^5 \tag{181}$$

which together with (146) yields

$$\epsilon_a{}^b \chi_b = -\chi\gamma_a\gamma^5 + \lambda_a\gamma^5. \tag{182}$$

For the sake of brevity we often introduce the obvious notations

$$\chi^2 = \chi\bar{\chi}, \quad \lambda^2 = \lambda^a\bar{\lambda}_a. \tag{183}$$

Other convenient identities used for λ_a in our present work are

$$\begin{aligned} (\lambda_a\bar{\lambda}_b) &= \frac{1}{2}\eta_{ab}\lambda^2, \\ (\lambda_a\gamma^5\bar{\lambda}_b) &= \frac{1}{2}\epsilon_{ab}\lambda^2, \\ (\lambda_a\gamma_c\bar{\lambda}_b) &= 0. \end{aligned} \tag{184}$$

The first of these identities can be proved by inserting the unit matrix $\gamma_a\gamma^a/2$ inside the product and interchanging the indices due to equation (180). The second and third equation is antisymmetric in indices a, b, and therefore to be calculated easily because they are proportional to ϵ_{ab}.

Quadratic combinations of the vector-spinor field can be decomposed in terms of irreducible components:

$$\begin{aligned} (\chi_a\bar{\chi}_b) &= \eta_{ab}\left(-\chi^2 + \frac{1}{2}\lambda^2\right) + 2(\chi\gamma_a\bar{\lambda}_b) \\ (\chi_a\gamma^5\bar{\chi}_b) &= \epsilon_{ab}\left(\chi^2 + \frac{1}{2}\lambda^2\right) \\ (\chi_a\gamma_c\bar{\chi}_b) &= 2\epsilon_{ab}(\chi\gamma^5\bar{\lambda}_c) \\ (\chi_a\gamma_c\gamma^5\bar{\chi}_b) &= 2\epsilon_{ab}(\chi\bar{\lambda}_c) \end{aligned} \tag{185}$$

B Anholonomic Basis in Two Dimensions

Purely bosonic two-dimensional space-time is best described in terms of the anholonomic orthonormal basis

$$e_a = \partial_a = e_a{}^m \partial_m \tag{186}$$

for tangent vectors. Many geometric quantities take a particular simple form involving the anholonomicity coefficients ${c_{ab}}^c$ defined by the commutator of the basis of vector fields

$$[e_a, e_b] = {c_{ab}}^c e_c. \tag{187}$$

From (186) they are expressed by the inverse zweibein

$$\begin{aligned} {c_{ab}}^c &= \left({e_a}^m \partial_m {e_b}^n - {e_b}^m \partial_m {e_a}^n\right) {e_n}^c \\ &= -{e_a}^m {e_b}^n \left(\partial_m {e_n}^c - \partial_n {e_m}^c\right). \end{aligned} \tag{188}$$

In two dimensions the anholonomicity coefficients are in one-to-one correspondence with their own trace

$$c_b = {c_{ab}}^a, \quad {c_{ab}}^c = {\delta_a}^c c_b - {\delta_b}^c c_a. \tag{189}$$

Eq. (188) shows that the anholonomicity coefficients are invariant under general coordinate transformations. Under the local Lorentz boost with parameter $\omega(x)$ they transform like a connection

$$\delta c_b = -\omega {\epsilon_b}^c c_c - {\epsilon_b}^c \partial_c \omega. \tag{190}$$

Some useful relations are

$$\epsilon^{ab} \partial_a {e_b}^m = \epsilon^{mb} c_b, \quad \epsilon^{mn} \partial_m {e_n}^a = -\epsilon^{ab} c_b. \tag{191}$$

Here the transformation from holonomic to anholonomic indices is performed by using the two-dimensional zweibein (not the supervielbein).

Apart from the zweibein a two-dimensional space-time is characterized by the Lorentz connection which may be written in anholonomic coordinates

$$\omega_a = {e_a}^m \omega_m. \tag{192}$$

Then the two-dimensional curvature tensor,

$$\widehat{R}_{mna}{}^b = (\partial_m \omega_n - \partial_n \omega_m) {\epsilon_a}^b, \tag{193}$$

yields the scalar curvature

$$\widehat{R} = 2\epsilon^{ab} \partial_a \omega_b + 2\epsilon^{ab} c_a \omega_b. \tag{194}$$

In two dimensions the trace of the torsion tensor $\hat{t}_b = {\hat{t}_{ab}}{}^a$ determines the full torsion tensor

$$\hat{t}_{ab}{}^c = {\delta_a}^c \hat{t}_b - {\delta_b}^c \hat{t}_a \tag{195}$$

and in the anholonomic basis becomes

$$\hat{t}_b = -c_b + \epsilon_b{}^c \omega_c. \tag{196}$$

Both zweibein and Lorentz connection are independent variables. For a given zweibein one can always construct a second Lorentz connection and other geometrical quantities corresponding to zero torsion. The latter condition makes the Lorentz connection depend on the zweibein and its first derivatives through the coefficients of anholonomicity (188)–(189)

$$\tilde{\omega}_a = \epsilon_a{}^b c_b. \tag{197}$$

Here and everywhere else the tilde sign means that the corresponding geometric quantity is derived for zero torsion. The difference between the two connections is given by the torsion tensor

$$\omega_a = \tilde{\omega}_a + \epsilon_a{}^b \hat{t}_b. \tag{198}$$

From (194) the scalar curvature corresponding to zero torsion can be expressed through the anholonomicity coefficients as well:

$$\widetilde{R} = 2\partial_a c^a + 2c_a c^a \tag{199}$$

We use covariant derivatives $\widehat{\nabla}_a = e_a{}^m \widehat{\nabla}_m = e_a{}^m(\partial_m + \omega_m)$ and $\widetilde{\nabla}_a = e_a{}^m \widetilde{\nabla}_m = e_a{}^m(\partial_m + \tilde{\omega}_m)$ for both types of Lorentz connections denoting the torsionless case by the tilde sign. They are simply related due to equation (198) and used alternatively according to the environment.

A sample relation exists between the two scalar curvatures:

$$\widehat{R} = \widetilde{R} + 2\widetilde{\nabla}_a \hat{t}^a = \widetilde{R} + 2\widehat{\nabla}_a \hat{t}^a - 2\hat{t}_a \hat{t}^a \tag{200}$$

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References

[1] J. Wess and B. Zumino, Phys. Lett. 66B (1977) 361 and Phys. Lett. 74B (1978) 51;
Yu. A. Gol'fand and E. P. Likhtman, JETP Lett. 13 (1971) 323;
D. V. Volkov and V. P. Akulov, Phys. Lett. B46 (1973) 109.

[2] D. Z. Freedman, P. van Nieuwenhuizen and S. Ferrara, Phys. Rev. D13 (1976) 3214;
D. Z. Freedman and P. van Nieuwenhuizen, Phys. Rev. D14 (1976) 912;
S. Deser and B. Zumino, Phys. Lett. 62B (1976) 335 and Phys. Lett. 65B (1976) 369;
R. Grimm, J. Wess and B. Zumino, Phys. Lett. 73B (1978) 415.

[3] C. Vafa, Nucl. Phys. B469 (1996) 403.

[4] E. Witten, Nucl. Phys. B443 (1995) 85.

[5] E. Sezgin, Phys. Lett. B392 (1997) 323.

[6] P. S. Howe, J. Phys. A: Math. Gen. 12 (1979) 393.

[7] P. Fayet, Phys. Lett. 69B (1977) 489.

[8] A. Hindawi, B. A. Ovrut and D. Waldram, Nucl. Phys. B471 (1996) 409.

[9] P. C. Aichelburg and T. Dereli, Phys. Rev. D18 (1978) 1754;
P. C. Aichelburg, Phys. Lett. 91B (1980) 382;
P. C. Aichelburg and R. Gueven, Phys. Rev. Lett. 51 (1983) 1613;
M. Rosenbaum, M. Ryan, L. F. Urrutia and R. Matzner, Phys. Rev. D34 (1986) 409;
M. E. Knutt-Wehlau and R. B. Mann, Nucl. Phys. B514 (1998) 355.

[10] M. O. Katanaev and I. V. Volovich, Phys. Lett. B175 (1986) 413 and Ann. Phys. 197 (1990) 1.

[11] P. Schaller and T. Strobl, Class. Quantum Grav. 11 (1994) 331 and Mod. Phys. Lett. A9 (1994) 3129;
T. Strobl, Phys. Rev. D50 (1994) 7346.

[12] T. Klösch and T. Strobl, Class. Quantum Grav. 13 (1996) 965, erratum ibid. 14 (1997) 825, Class. Quantum Grav. 13 (1996) 2395 and Class. Quantum Grav. 14 (1997) 1689.

[13] M. O. Katanaev, W. Kummer and H. Liebl, Phys. Rev. D53 (1996) 5609 and Nucl. Phys. B486 (1997) 353.

[14] M. O. Katanaev, J. Math. Phys. 34 (1993) 700 and J. Math. Phys. 38 (1997) 946.

[15] W. Kummer, in *Hadron Structure '92*, edited by D. Brunsko and J. Urbán, Košice University, (1992) pp. 48–56.

[16] S. N. Solodukhin, Phys. Lett. B319 (1993) 87 and Phys. Rev. D51, no. 2 (1995) 603.

[17] W. Kummer and D. J. Schwarz, Nucl. Phys. B382 (1992) 171.

[18] F. Haider and W. Kummer, Int. J. Mod. Phys. A9 (1994) 207.

[19] W. Kummer, H. Liebl and D. V. Vassilevich, Nucl. Phys. B513 (1998) 723.

[20] G. Mandal, A. Sengupta and S. R. Wadia, Mod. Phys. Lett. A6 (1991) 1685;
S. Elitzur, A. Forge and E. Rabinovici, Nucl. Phys. B359 (1991) 581;
E. Witten, Phys. Rev. D44 (1991) 314;
C. G. Callan Jr., S. B. Giddings, J. A. Harvey and A. Strominger, Phys. Rev. D45 (1992) 1005.

[21] A. Fabbri and J. G. Russo, Phys. Rev. D53 (1996) 6995.

[22] V. S. Vladimirov and I. V. Volovich, Theor. Math. Phys. 59 (1984) 317 and Theor. Math. Phys. 60 (1985) 743.

[23] R. Haag, J. T. Lopuszanski and M. Sohnius, Nucl. Phys. B88 (1975) 257.

[24] P. van Nieuwenhuizen, Phys. Rep. 68 (1981) 189.

[25] N. Ikeda, Ann. Phys. 235 (1994) 435.

[26] F. A. Berezin, The Method of Second Quantization (Academic Press, New York and London, 1966).

[27] B. DeWitt, Supermanifolds, Cambridge Monographs on Mathematical Physics (Cambridge University Press, 1984).

[28] S. Nojiri and I. Oda, Mod. Phys. Lett. A8 (1993) 53;
Y. Park and A. Strominger, Phys. Rev. D47 (1993) 1569;
A. Bilal, Phys. Rev. D48 (1993) 1665.

[29] B. M. Barbashov, V. V. Nesterenko and A. M. Chervjakov, Theor. Math. Phys. 40 (1979) 572;
R. Jackiw, in *Quantum Theory of Gravity. Essays in Honor of the 60th Birthday of Bryce S. DeWitt*, edited by S. Christensen (Hilger, Bristol, 1984) pp. 403–420;
C. Teitelboim, in *Quantum Theory of Gravity. Essays in Honor of the 60th Birthday of Bryce S. DeWitt*, edited by S. Christensen (Hilger, Bristol, 1984) pp. 327–344.

[30] J. F. Arvis, Nucl. Phys. B212 (1983) 151;
E. Martinec, Phys. Rev. D28 (1983) 2604.

[31] J. N. G. N. Prata, Phys. Lett. B405 (1997) 271;
L. Zhao and C. Qu, Int. J. Theor. Phys. 36 (1997) 1537.

[32] P. Nelson and G. Moore, Nucl. Phys. B274 (1986) 509;
S. Randjbar-Daemi, A. Salam and J. Strathdee, Nucl. Phys. B320 (1989) 221.